

B-Model Approach to Instanton Counting

Daniel Krefl^a and Johannes Walcher^b

^a *Center for Theoretical Physics*

Seoul National University, Seoul, South Korea

^b *Department of Physics, and Department of Mathematics and Statistics*

McGill University, Montreal, Quebec, Canada

Abstract

The instanton partition function of $\mathcal{N} = 2$ gauge theory in the general Ω -background is, in a suitable analytic continuation, a solution of the holomorphic anomaly equation known from B-model topological strings. The present review of this connection is a contribution to a special volume on recent developments in $\mathcal{N} = 2$ supersymmetric gauge theory and the 2d-4d relation, edited by J. Teschner.

December 2014

1 Introduction and Key Ideas

The instanton partition function of $\mathcal{N} = 2$ supersymmetric quantum field theories

$$Z^{\text{inst}}(a, \epsilon_1, \epsilon_2; \Lambda) \tag{1.1}$$

is of algebra-geometro-physical interest for at least three different, though related, reasons. First of all, by its very definition, Z^{inst} encapsulates the cohomology of the moduli space of instantons, supersymmetric solutions of the underlying classical field theory, and the algebraic structures on that space (chapter [V:3]). Secondly, within the 2d-4d correspondences of Alday-Gaiotto-Tachikawa, and Nekrasov-Shatashvili (chapter [V:11]), the instanton partition function connects supersymmetric field theories with the world of completely integrable systems and their quantization, specifically Hitchin systems (see chapter [V:2]). Thirdly, $Z^{\text{inst}}(a, \epsilon_1, \epsilon_2; \Lambda)$ contains information about the structure of the Coulomb branch that goes beyond the weakly coupled description in a Lagrangian field theory. After a suitable analytic continuation, it allows to calculate interesting physical quantities everywhere in the moduli space of vacua and marginal couplings, and thereby to study a variety of dualities.

It is in fact, this latter aspect of the instanton partition function that is closest to the approach pioneered by Seiberg and Witten for solving the low-energy dynamics of $\mathcal{N} = 2$ supersymmetric field theories, by exploiting the global constraints on the structure of the moduli space coming from special geometry and modular invariance. The basic ideas are easily explained.

The instanton partition function Z^{inst} calculated via localization (see chapter [V:3]) is a series

$$Z^{\text{inst}} \sim \sum_n \Lambda^n R_n(a, \epsilon_1, \epsilon_2), \tag{1.2}$$

with rational functions R_n of the Ω -background parameters $\epsilon_{1,2}$, and the Coulomb branch parameters a , that converges well for small instanton counting parameter Λ . As explained by Nekrasov [1], the Seiberg-Witten solution for the low-energy effective action is recovered in the non-equivariant limit $\epsilon_{1,2} \rightarrow 0$. Specifically, the $\mathcal{N} = 2$ prepotential is the residue

$$\mathcal{F}^{(0)}(a; \Lambda) = \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} (\epsilon_1 \epsilon_2 \log Z^{\text{inst}}(a, \epsilon_1, \epsilon_2; \Lambda)), \tag{1.3}$$

after the perturbative expansion of the free energy for small Λ (in the asymptotically free case, this is equivalent to the weak-coupling limit $a \rightarrow \infty$). It coincides with the

prepotential obtained from the Seiberg-Witten effective geometry (the family of hyper-elliptic curves together with the differential), which captures the low-energy dynamics and is the basis for the various embeddings into string theory. Most specifically, the $\mathcal{F}^{(0)}$ appears in the so-called geometric engineering limit of the prepotential governing the compactification of type II string on a (non-compact) Calabi-Yau manifold [2].

It has been a natural question to ask for the analytic and geometric characterization of the terms in Z^{inst} of higher order in ϵ_1, ϵ_2 , and their physical interpretation. As anticipated already by Nekrasov [1], the answer is most immediate on the special slice in coupling constant space $\epsilon_1 = -\epsilon_2$. By a detour in one higher dimension, one can see that in general, the Ω -background in the gauge theory arises in the string/M-theory constructions from a vacuum expectation value of the gravi-photon field strength, of the form

$$F = \epsilon_1 dx_1 \wedge dx_2 + \epsilon_2 dx_3 \wedge dx_4 \quad (1.4)$$

(itself a limit of the Melvin background, or “flux-trap” [3] in other duality frames). The specialization $\epsilon_1 = -\epsilon_2$ corresponds to a self-dual gravi-photon background, and the expansion coefficients of the supersymmetric free energy in $\epsilon_1 = -\epsilon_2 =: g_s$ are identified with the higher-derivative F-term couplings $R^2 F^{2g-2}$ in the effective action [4], which can be computed as the *topological* string genus- g free energy [5]. In the geometric engineering limit, one recovers the expansion of the instanton partition function.

$$\log Z^{\text{inst}}(a, g_s, -g_s; \Lambda) = \sum_{g=0}^{\infty} g_s^{2g-2} \mathcal{F}^{(g)}(a) \quad (1.5)$$

A natural way to test this physical interpretation is to re-calculate the $\mathcal{F}^{(g)}$ using the string theory methods. In the topological B-model, the most universal of these methods is the holomorphic anomaly of BCOV [5].

The basic message of the holomorphic anomaly method is that the higher order corrections $\mathcal{F}^{(g)}(a)$ can still be continued throughout moduli space, in particular any strong coupling regions, but (in distinction to the prepotential $\mathcal{F}^{(0)}$), they are no longer holomorphic functions of a . The physical origin of this non-holomorphicity are the infrared effects, degenerating Riemann surfaces in the perturbative string theory [5], or the distinction between 1 PI and Wilsonian effective action from the point of view of the field theory [6]. Mathematically, the holomorphic anomaly is an expression of the competition between holomorphy and modular invariance [5], and can also be viewed as an embodiment of the wave-function nature of the topological partition function [7].

The holomorphic anomaly *equation* dictates the non-holomorphic dependence of $\mathcal{F}^{(g)}(a, \bar{a})$ recursively in the order of the expansion, $2g - 2$. The meromorphic function on moduli space that is thereby left undetermined at each order is known as the holomorphic ambiguity and can, under favorable circumstances, be determined by imposing appropriate principal parts or “boundary conditions” at the various singular points.

It was shown by Klemm and Huang [8] that the holomorphic anomaly commutes with the geometric engineering limit, and can be used to completely recover the $\mathcal{F}^{(g)}$ in the expansion (1.5). Even though a detailed derivation of the holomorphic anomaly from the gauge theory point of view is missing, the equation itself can be written down based solely on the special geometry data on the moduli space that can be obtained from the prepotential $\mathcal{F}^{(0)}$. And at least in all examples with low-dimensional moduli space, the boundary conditions at the monopole/dyon points are sufficient to completely fix the holomorphic ambiguity, and thereby make the holomorphic anomaly “integrable” in that sense.

From the point of view of instanton counting (1.1), this discussion of the holomorphic anomaly appears as rather tangential. After all, the holomorphy and integrability of the higher order corrections is built into the formalism, while the underlying spectral geometry is completely determined by the first order classical term, *i.e.*, the prepotential. There are nevertheless several very good reasons to explore the connection further, and in particular, to understand the extension of the holomorphic anomaly to the full two-parameter Ω -background, away from the specialization $\epsilon_1 = -\epsilon_2$.

From the gauge theory point of view, the precise role of the holomorphic anomaly, or the wave-function nature of Z^{inst} is not completely understood, for instance in the context of the quantum integrable systems. Moreover, the possible calculation of Z^{inst} as a “sum over instantons” (or some other semi-classical configurations) around other points in moduli space remains to be explored. While conformal field theory in principle provides formal expressions for Z^{inst} in terms of certain contour integrals also elsewhere in moduli space, these have been evaluated explicitly only in a limited number of situations. The continuation of the $\mathcal{F}^{(g)}$ to other points in moduli space via the holomorphic anomaly provides a very welcome benchmark for such calculations.

From the string theory point of view, the existence of the second deformation parameter itself is the most intriguing aspect. Indeed, while the role of $g_s = \sqrt{-\epsilon_1 \epsilon_2}$ as the genus-counting parameter, *i.e.*, the topological string coupling constant, is readily appreciated, the existence of a second “string-coupling like” parameter is much more

mysterious. Since, in much more generality than the restricted topological context, string theory does not have any free parameters, the absence of a worldsheet description would be in tremendous tension with the overall picture. To be sure, the role of the second parameter from the macroscopic space-time, or M-theory point of view is completely clear, see [1, 9], as well as the connection with refinement and categorification [10]. What is missing is the *microscopic* explanation.

The main point of the present contribution is to highlight the observation that with the right choice of parameterization of the coupling constants, the deformation away from the special slice $\epsilon_1 = -\epsilon_2$ is indeed as simple as it could be: The higher order corrections for general ϵ_1, ϵ_2 still satisfy the holomorphic anomaly equations, with deformation only in the boundary conditions. In particular, a single infinitesimal coupling constant is sufficient. This was first pointed out in [11, 12, 13]. These methods therefore allow the calculation (via “analytic” continuation) of Z^{inst} around points in moduli space other than the weak coupling regime. This constitutes a benchmark for testing the 2d-4d relation this special volume is about away from $\Lambda \rightarrow 0$. Coming back to string theory, these observations have allowed the application of the holomorphic anomaly equation for the B-model calculation of refined BPS invariants of local Calabi-Yau manifolds [14, 15, 16]. This can be viewed as further evidence that the second parameter should be lifted to the topological string (not necessarily as a coupling constant, but rather as a deformation parameter), and has been as well applied and interpreted in the context of quantum geometry and quantum integrability [17]. Among the possible stringy explanations of the refinement, we will outline in somewhat more detail an intriguing relation to orbifolds and orientifolds, following [11, 18].

Before closing this introduction, it seems worthwhile to emphasize once again that in this Chapter, we are discussing the instanton partition function from the point of view of the “B-model”, meaning the global structure of the moduli space, special geometry and modular invariance. In contrast to (1.2), which is exact in ϵ_1, ϵ_2 , but perturbative in Λ , the B-model provides answers that are exact in the instanton expansion, but perturbative in ϵ_1, ϵ_2 .

2 Geometric Engineering

Large classes of supersymmetric gauge theories in various dimensions can be systematically obtained from string-, M- and F-theory compactifications. This is usually referred

to as geometric engineering, as the geometry of the compactification manifold X determines the effective gauge theory in the field theory limit. We only give a lightning overview, excellent pedagogical reviews being available in the canonical literature, see for instance, [19, 20].

Any given gauge theory can typically be realized in several ways in string theory. These different constructions are then related by various dualities and limiting procedures. Hence, depending on the gauge theory to be investigated via geometric engineering, and the specific gauge theory property under investigation, a convenient duality frame has to be chosen. A common feature of all geometric engineering approaches is that in order to decouple string and gravity effects, the compactification manifold X has to feature a local singularity, perhaps in the guise of a brane.

We are interested in $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions, their low-energy effective prepotential $\mathcal{F}^{(0)}(a, m)$, and higher derivative F-term couplings. These are (modulo the holomorphic anomaly) holomorphic functions of the Coulomb moduli a_i and masses of matter fields m_i , and receive their essential contributions from the space-time instantons. This class of theories can be conveniently engineered and investigated in a type IIA/B superstring framework by compactification on a local (non-compact) Calabi-Yau 3-fold, which yields, under certain conditions, a four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory with decoupled gravity.

To be specific, consider type IIA string theory compactified on a Calabi-Yau 3-fold X . Four-dimensional abelian gauge fields arise in the Ramond-Ramond sector by dimensional reduction on the even cohomology of X . But since perturbative string states do not carry Ramond-Ramond charge, in order to obtain interesting non-abelian gauge groups, we must include non-perturbative effects. In particular, D2-branes wrapped on the (compact) 2-cycles of the compactification geometry represent objects electrically charged under the corresponding abelian gauge fields. The masses of these states being proportional to the Kähler class (volume) t_{f_i} of the wrapped 2-cycles, one needs $t_{f_i} \rightarrow 0$ in order to have massless charged gauge bosons.

In fact, it is best to view the Calabi-Yau compactification as the dimensional reduction of a K3 compactification near an ADE singularity. The gauge group originates in six dimensions from the (compact) homology of the singularity (of ADE type for ADE gauge group), while further dimensional reduction (on a copy, $B \cong \mathbb{P}^1$, of complex projective space to be specific) leads to an $\mathcal{N} = 2$ gauge theory in four dimensions. In this process, the bare gauge coupling g_{YM} of the four-dimensional theory is pro-

portional to the Kähler class t_B of the 2-dimensional manifold used for the reduction, *i.e.*, $t_B \sim 1/g_{YM}^2$. In order to decouple gravity (and stringy effects) it is sufficient to send the coupling constant to zero, since it pushes the string scale to infinity [2]. This means that we are interested in the limit $t_B \rightarrow \infty$.

In order to satisfy the Calabi-Yau condition, the compactification space can not be given by a direct product, but rather must have the structure of a fibration,

$$F \rightarrow X \rightarrow B, \quad (2.1)$$

where the fiber geometry F (with the ADE singularity) determines the gauge group while the base geometry B the effective gauge coupling in four dimensions. Note that the fibration structure also allows to incorporate matter content via local enhancement of the fiber singularity.

It is important to keep in mind that the limits $t_B \rightarrow \infty$ and $t_{f_i} \rightarrow 0$ of base and fiber Kähler classes are not independent. This can be illustrated best at hand of a concrete example. Consider the geometric engineering of pure $SU(2)$ along the lines sketched above, as originally discussed in [2]. To obtain the two charged gauge bosons, W^+ and W^- , it is sufficient to fiber a \mathbb{P}^1 over the base \mathbb{P}^1 . [The different ways this can be done are labeled by an integer and the corresponding geometries correspond to the Hirzebruch surfaces. The Calabi-Yau 3-fold itself is the total space of the anti-canonical bundle over this complex surface. All Hirzebruch surfaces give rise to pure $SU(2)$ in four dimensions.] Recall that in the weak coupling regime the running of the gauge coupling is given by

$$\frac{1}{g_{YM}^2} \sim \log \frac{m}{\Lambda}, \quad (2.2)$$

where m denotes the mass of the W-bosons and Λ the dynamical scale. With the above identifications, we learn that we have to take the limit in a way such that $t_B \sim \log t_f$ holds. The precise proportionality constant can be fixed as follows: We know that at weak coupling the instanton corrections to the bare gauge coupling go in powers of $(\Lambda/a)^4$, with a the Coulomb modulus. Correspondingly, we have to scale $e^{-t_b} \sim \delta^4 \Lambda^4$ and $t_f \sim \delta a$ as $\delta \rightarrow 0$, which constitutes the map between the string moduli and the gauge theory parameters, for pure $SU(2)$.

The useful property of the type IIA string construction is that the space-time instanton corrections are mapped to world-sheet instanton corrections. Qualitatively, this is clear from the relation between the 2-cycles and the gauge coupling and gauge

bosons sketched above: The Euclidean string worldsheet is wrapped around the 2-cycles of the geometry with worldsheet instanton action $S \sim d_b t_b + d_f t_f$, where d_b and d_f refer to wrapping numbers. In particular, this means that we do not have to consider the full type IIA string theory to investigate the gauge theory from a string point of view. Rather, the topological sector is sufficient, *i.e.*, the topological string amplitudes which capture world-sheet instanton corrections.

Starting from the topological string tree-level amplitude, taking the above gauge theory limit yields the space-time instanton corrections to the gauge theory prepotential. The higher-genus amplitudes encode the gravitational corrections, as sketched in the introduction. In this way, the string theory provides both a conceptual framework, and a host of computational methods to investigate non-perturbative effects in supersymmetric gauge theories.

There is, however, one important subtlety to keep in mind. Since the geometric engineering limit involves $t_{f_i} \rightarrow 0$, the compactification geometry is in fact singular, and we are not expanding the string amplitudes around the large volume point in moduli space. Hence, if we compute the topological string amplitudes using the usual A-model techniques, which are valid at large volume (such as, localization [21] or the topological vertex [22]), these amplitudes have to be analytically continued before we can take the limit. [The necessity of this analytic continuation is quite clear already from the fact that the large volume expansion of the topological amplitudes is a series in the exponentiated Kähler moduli, whereas the gauge theory prepotential at weak coupling is an expansion into negative powers of the Coulomb branch parameter.]

For the topological string amplitudes of the $SU(2)$ engineering geometry sketched above, the analytic continuation can be achieved via a relatively simple resummation [23]. In more complicated examples, one has to switch to the B-model mirror of the type IIA string background in order to perform the analytic continuation. We recall that in general, mirror symmetry maps worldsheet instanton corrections to the expansions of classical geometric quantities. For instance, tree-level worldsheet instantons in type IIA are encoded in the period integrals of the type IIB mirror geometry. As these periods can be calculated as solutions of simple linear differential equations, their analytic continuation all over moduli space is straight-forward.

Under the geometric engineering limit along the lines reviewed above, mirror symmetry may be seen as the stringy origin of the Seiberg-Witten solution of $\mathcal{N} = 2$ gauge theory. That is, the Seiberg-Witten curve and differential arise in the limit of the mir-

ror Calabi-Yau threefold which is mirror dual to the type IIA engineering geometry. In particular, as its B-model parent, the Seiberg-Witten geometry naturally provides a global description of the moduli space.

So far we mainly had in mind spherical world-sheet instantons yielding instanton corrections to the gauge theory prepotential. However, perhaps the most useful property of this stringy construction is that it allows to calculate gravitational corrections to the $\mathcal{N} = 2$ gauge theory, originating from world-sheet instantons of higher genus. In detail, the genus- g topological string amplitude yields $R^2 F^{2g-2}$ corrections to the gauge theory [4], which can be calculated very efficiently via a specific topological string B-model technique, namely the holomorphic anomaly equation, all over the Coulomb moduli space. This is the subject to which we now turn.

3 B-model

It has been observed some time ago in [8] that the free energy of four dimensional $\mathcal{N} = 2$ supersymmetric gauge theory with gravitational corrections satisfies the holomorphic anomaly equations of [24, 5]. This can be seen as a consequence of the geometric engineering approach to $\mathcal{N} = 2$ gauge theories, where the gauge theory free energy follows as a specific limit of the topological string free energy on a corresponding engineering Calabi-Yau, as outlined in the previous section.

Since the gravitational corrections captured by the topological string are a specific specialization of the Ω -deformed gauge theory, $\epsilon_1 = -\epsilon_2$, it is natural to ask whether the Ω -deformed theory with general equivariant parameters satisfies as well a kind of anomaly equation. The main point of interest is that the holomorphic anomaly equation allows to analytically continue the Ω -deformed partition function over all of the Coulomb moduli space. In contrast, the instanton counting partition function of [1] and as well the CFT calculations for the partition function via the AGT correspondence [25] are most useful only for the asymptotically free theories at a weakly coupled point in Coulomb moduli space (see however [26]).

The complete partition function $Z^{\text{inst}}(\epsilon_1, \epsilon_2)$ obtained via instanton counting is exact in the two equivariant parameters $\epsilon_{1,2}$. In order to get started with the investigation of the anomaly equation, one has to choose a parameterization of the infinitesimal neighborhood of $\epsilon_{1,2} = 0$, and the form of the answer will naively depend on the choice. It turns out that the correct expansion from the topological string point of view is to

choose the same parameterization as occurring in the AGT correspondence, used in this context in [11, 13]. Namely, we write

$$\epsilon_1 = \sqrt{\beta} g_s, \quad \epsilon_2 = -\frac{1}{\sqrt{\beta}} g_s, \quad (3.1)$$

with β a fixed constant and g_s being the only infinitesimal expansion parameter. Hence (leaving the Coulomb moduli a implicit) we define the perturbative amplitudes $\mathcal{F}^{(g)}(\beta)$ as the coefficients in the expansion

$$\log Z^{\text{inst}}(\epsilon_1, \epsilon_2) = \mathcal{F}(\epsilon_1, \epsilon_2) = \sum_{g=0}^{\infty} \mathcal{F}^{(g)}(\beta) g_s^{2g-2}. \quad (3.2)$$

We note in particular that the Seiberg-Witten prepotential defined via the limit (1.3), is *independent* of β , *i.e.*,

$$\mathcal{F}(\epsilon_1, \epsilon_2) = \mathcal{F}^{(0)} g_s^{-2} + \mathcal{O}(g_s^0). \quad (3.3)$$

Of course, one might also envisage a double expansion in the two-parameters ϵ_1, ϵ_2 , as performed in this context in [12]. However, it is not hard to see that the two-parameter expansion is related via a finite resummation to the one-parameter expansion (3.2). This is related to the fact that the $\mathcal{F}^{(g)}(\beta)$ are *polynomial* in β . As a consequence, an anomaly equation for a two-parameter expansion scheme is algebraically equivalent with the anomaly for the one-parameter expansion (*cf.*, the discussions in [27, 28]). Our results, and specifically, the fact that only the holomorphic ambiguity depends on β , make it clear that the one-parameter expansion (3.2) is most economical and therefore preferred.¹

Note that with the four-dimensional Lorentz invariance, the expansion (3.2) goes in even powers of g_s only, reflecting the symmetry $\epsilon_{1,2} \rightarrow -\epsilon_{1,2}$ of the Ω -background. As noted in [1, 29], localization in the presence of mass parameters in principle can violate this symmetry, and odd powers of g_s will be present as well. However, this odd sector is not fundamental, and can be “gauged away” by a linear shift of the appropriate mass parameters. Notably, this does not apply to the theories in the presence of additional extended objects like surface operators (discussed in chapter [V:7]). Such a setup breaks 4-d Lorentz invariance and a true odd sector in g_s will be generated. We will here only consider the even in g_s case, while emphasizing that the general case can be

¹This is not to say that the microscopic origin of the holomorphic anomaly might not be better explained in the two-parameter scheme, see [28].

treated as well [11], by appealing to the extended holomorphic anomaly equations of [30, 31].

So let us now explain in detail the method of the holomorphic anomaly for the calculation of the $\mathcal{F}^{(g)}(\beta)$. We begin with the role of $\mathcal{F}^{(0)}$ in special geometry. We denote by u a global coordinate on the moduli space \mathcal{M} of vacua, which is identified with the base space of an appropriate family of complex curves, \mathcal{C}_u . (For simplicity, we will write equations only in the case that \mathcal{M} is one-dimensional. The reader might have in mind $SU(2)$ Seiberg-Witten theory with $N_f < 4$ fundamental flavors. Some aspects of the higher-rank theory are discussed in [12, 15].) The family of curves is equipped with a meromorphic one-form λ_{SW} , such that for appropriate choice of one-cycles A and A_D on \mathcal{C}_u , the periods

$$a = \oint_A \lambda_{\text{SW}}, \quad a_D = \oint_{A_D} \lambda_{\text{SW}}, \quad (3.4)$$

satisfy the relation

$$a_D = \frac{\partial \mathcal{F}^{(0)}}{\partial a}, \quad (3.5)$$

after eliminating u from (3.4). We do not need to be explicit about this auxiliary geometric data, for which we refer to chapter [V:1]. However, one should keep in mind, as already mentioned in the previous section, that this auxiliary data originates from the mirror Calabi-Yau geometry of the corresponding geometric engineering geometry.

For expansion in different regions of moduli space, it is most convenient to base the development on the Picard-Fuchs equation, a third order system of linear differential equations,

$$\mathcal{L}\varpi(u) = 0, \quad (3.6)$$

satisfied by all periods of λ_{SW} . Using a as a local coordinate around $u \rightarrow \infty$, the Picard-Fuchs operator takes the form²

$$\mathcal{L} = \partial_a \frac{1}{C_{aaa}} \partial_a^2, \quad (3.7)$$

where

$$C_{aaa} = \partial_a^3 \mathcal{F}^{(0)} = \partial_a^2 a_D(a) = \partial_a \tau(a), \quad (3.8)$$

is a (meromorphic) rank three symmetric tensor over \mathcal{M} , which in the topological string context is referred to as the Yukawa coupling, and $\tau(a)$ corresponds to the

²As is now evident, the constant is a third solution of the differential equation. This solution decouples in special cases, such as $SU(2)$ gauge theory with massless hypermultiplets.

complexified effective gauge coupling. In particular, $g \sim \text{Im}\tau$ is the Weil-Petersson (or σ -model) metric on \mathcal{M} , which plays a central role in special geometry on \mathcal{M} . Another important feature is the existence of canonical (flat) coordinates [5], which provide a meaningful expansion parameter around any interesting point $u = u_*$ in \mathcal{M} . In such a flat coordinate $t = t(u)$, vanishing at $u = u_*$, the Picard-Fuchs operator takes again the form (3.7) with $a \rightarrow t$, *i.e.*,

$$\mathcal{L} = \partial_t \frac{1}{C_{ttt}} \partial_t^2, \quad C_{ttt} = \left(\frac{\partial u}{\partial t} \right)^3 C_{uuu}. \quad (3.9)$$

We are now ready to write down the holomorphic anomaly equations of [5]. Recall that the specialization of the gauge theory amplitude $\mathcal{F}^{(g)}(\beta)$ to $\beta = 1$, (namely, the self-dual background $\epsilon_1 = -\epsilon_2$) arises via geometric engineering from the genus- g topological string amplitude. The statement of BCOV is that the topological string amplitudes, while holomorphic in the Kähler moduli, are not well-behaved globally over the moduli space. Instead, one should view the topological string amplitudes as a holomorphic limit of non-holomorphic, but globally defined objects. Under the gauge theory limit sketched in the previous section, this translates to the statement that one should view the gauge theory $\mathcal{F}^{(g)}(a)$ (for $g \geq 1$) as the holomorphic limit $\bar{a} \rightarrow \infty$ of *non-holomorphic, but globally defined* objects $\mathcal{F}^{(g)}(u, \bar{u})$, arising from the topological string amplitudes in the gauge theory limit. (These are customarily denoted by the same letter, as confusion can not arise.) Similarly, the holomorphic anomaly equation satisfied by the topological string amplitudes translates to a recursive relation for the gauge theory $\mathcal{F}^{(g>1)}(u, \bar{u})$, *i.e.*,

$$\bar{\partial}_{\bar{u}} \mathcal{F}^{(g)} = \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ g_i > 0}} \bar{C}_{\bar{u}}^{uu} \mathcal{F}_u^{(g_1)} \mathcal{F}_u^{(g_2)} + \frac{1}{2} \bar{C}_{\bar{u}}^{uu} \mathcal{F}_{uu}^{(g-1)}, \quad (3.10)$$

where $\mathcal{F}_{uu}^{(g)} = D_u \mathcal{F}_u^{(g)} = D_u^2 \mathcal{F}^{(g)}$, D_u is the covariant derivative over \mathcal{M} , and indices are raised and lowered using the Weil-Petersson metric. The holomorphic limit of the connection of the Weil-Petersson metric (entering the covariant derivative) on \mathcal{M} takes the simple form

$$\lim_{\bar{t} \rightarrow 0} \Gamma_{uu}^u = \partial_u \log \frac{\partial t(u)}{\partial u}. \quad (3.11)$$

The “one-loop” amplitude satisfies the special equation

$$\bar{\partial}_{\bar{u}} \partial_u \mathcal{F}^{(1)} = \frac{1}{2} \bar{C}_{\bar{u}}^{uu} C_{uuu}. \quad (3.12)$$

At the level of the topological string, the holomorphic anomaly originates from topological anomalies, that is, under coupling to gravity (including integration over moduli of Riemann surfaces), the theory is only “almost” topological, as there are contributions from the boundaries of moduli spaces of genus- g Riemann surfaces to certain topologically trivial correlator insertions. More specifically, a genus- g Riemann surface can degenerate either to a genus- $(g - 1)$ surface with two extra punctures via pinching of a handle, or to two disconnected surfaces with an extra puncture of genus g_a and g_b (with $g = g_a + g_b$) via pinching of a tube. These two boundary contributions are reflected in the holomorphic anomaly equation (3.10). Although the topological string origin of the anomaly equation (and in particular of $\mathcal{F}^{(g)}(u, \bar{u})$) is clear, less so is the precise supergravity (or gauge theory) meaning and/or origin thereof. Hence, the main justification of (3.10) at the level of gauge theory comes as a limit of the topological string via geometric engineering. However, an independent justification can be given along the lines of Witten’s wave-function interpretation of the topological string partition function [7]. The reasoning leading to this interpretation of the $\mathcal{F}^{(g)}$ and the recursive relations between them relies solely on the special geometry (the holomorphic symplectic structure) of the moduli space (viewed as a classical phase space). Starting from the Seiberg-Witten geometry, this reasoning can therefore also be applied directly to the gauge theory. We will come back to this interpretation in section 5.

The natural question to ask is how (3.10) should be modified away from $\beta = 1$. The answer provided by [11] is the simplest possible *not at all!* More precisely, in [11], the use of the localization formulas of [1] resulted in the presence of non-vanishing terms of odd order in g_s , suggesting a role for the extended holomorphic anomaly equation of [30, 31], as well as a relation to topological string orientifolds. The extension data (the term at order g_s^{-1}) was also identified in simple geometric terms on the Seiberg-Witten curve. In [13] it was observed that the shift of the mass parameters [29] removes those odd terms, as mentioned above. While it is reassuring to see that the formalism works well with either prescription, we here only present the shifted version, as it is more economical.

The content of the equations (3.10), (3.12) is that due to the anti-holomorphic derivative, the $\mathcal{F}^{(g)}$ are determined up to holomorphic terms, the so-called holomorphic ambiguity. The standard technique to fix this ambiguity is by taking known characteristics of the $\mathcal{F}^{(g)}$ at specific points in moduli space as boundary conditions into account. For instance, topological string amplitudes expanded near a point in moduli

space where the target space develops a conifold singularity show a characteristic “gap” structure [8, 32] (we denote the modulus of the deformation as t_c)

$$\mathcal{F}^{(g>1)} = \Psi^{(2g-2)}(1) t_c^{-2g+2} + \mathcal{O}(t_c^0), \quad (3.13)$$

with leading non-vanishing coefficients $\Psi^{(g)}(1)$ of the singular terms given by the free energy of the $c = 1$ string at the self-dual radius $R = 1$ [33] (we use here a normalization different from the one usually used in the CFT context). Knowledge of the conifold expansion (3.13) is usually sufficient to fix the holomorphic ambiguity to very high genus [32], or, even to fix it completely [34], depending on the specific model. The coefficient of the singular term in (3.13), $\Psi^{(n)}(1)$, can be seen as due to integrating out a single massless hypermultiplet in the effective action [35] and is therefore rather universal.³ In particular, expansion of the gauge theory free energy near a point in moduli space with a massless monopole/dyon (hypermultiplet) should show the same behavior, and indeed does [8].

In the generalization to arbitrary $\beta \neq 1$ we then must have that the boundary conditions are not given by integrating out a massless hypermultiplet in an anti-selfdual background, but rather in the Ω -background. Hence, the coefficients $\Psi^{(g)}(1)$ change to β -dependent functions $\Psi^{(n)}(\beta)$ captured by the Schwinger type integral

$$\mathcal{F}_{c=1}(\epsilon_1, \epsilon_2; t_c) := \int \frac{ds}{s} \frac{e^{-t_c s}}{4 \sinh\left(\frac{\epsilon_1 s}{2}\right) \sinh\left(\frac{\epsilon_2 s}{2}\right)} \sim \dots + \sum_{n>0} \Psi^{(n)}(\beta) \left(\frac{g_s}{t_c}\right)^n,$$

under the usage of (3.1). Interestingly, the free energy $\mathcal{F}_{c=1}(\epsilon_1, \epsilon_2; t_c)$ still corresponds to the $c = 1$ string free energy, albeit at general radius $R = \beta$. The corresponding partition function is also known as Gross-Klebanov partition function, and we have for the expansion coefficients the following closed expressions [36]

$$\begin{aligned} \Psi^{(0)}(\beta) &= -\frac{1}{24} \left(\beta + \frac{1}{\beta} \right), \\ \Psi^{(n)}(\beta) &= (n-1)! \sum_{k=0}^{n+2} (-1)^k \frac{B_k B_{n+2-k}}{k!(n+2-k)!} (2^{1-k} - 1)(2^{k-n-1} - 1) \beta^{k-n/2-1}. \end{aligned} \quad (3.14)$$

Using the coefficients (3.14) as boundary conditions for general (real) β and analytically continuing back to the weakly coupled regime, somewhat surprisingly reproduces

³It that sense, the singularity structure (but not the regular terms) in those strong coupling regions does follow from a field theory computation.

for $SU(2)$ with massless $N_f < 4$ flavors the instanton counting results of [1] (after appropriate choice of gauge of mass parameters, *cf.*, discussion above), as first reported in [11, 13].

Similarly, using (3.14) as boundary conditions for the topological string expanded near a conifold singularity of specific local Calabi-Yau geometries reproduces under analytic continuation the refined free energy defined via the 5d instanton counting. However, there is one important subtlety, which is usually not explicitly mentioned in the literature. Namely, even for a simple Calabi-Yau like local \mathbb{P}^2 or $\mathbb{P}^1 \times \mathbb{P}^1$, the boundary conditions (3.14) alone are not sufficient to completely fix the holomorphic ambiguity. The actual difference comes in at 1-loop. Generally, the 1-loop holomorphic ambiguity possesses not only a contribution from the conifold discriminant, but also from the large volume divisor. For example, the 1-loop ambiguity $a^{(1)}(\beta)$ of refined local \mathbb{P}^2 reads [17]

$$a^{(1)}(\beta) = \Psi^{(0)}(\beta) \log \Delta + \kappa(\beta) \log z, \quad (3.15)$$

with Δ parameterizing the conifold locus, *i.e.*, $\Delta := (1 - 27z)$ and

$$\kappa(\beta) = -\Psi^{(0)}(\beta) - \frac{2}{3}. \quad (3.16)$$

Note that in contrast to $\Psi^{(n)}(\beta)$, we do not know how to infer $\kappa(\beta)$ from first principles. Rather $\kappa(\beta)$ has to be manually chosen appropriately to reproduce the desired 1-loop free energy model by model.

4 Refinement vs. Orbifolds

We observed in the previous section that refinement near a conifold point in moduli space can be interpreted as a radius deformation of the $c = 1$ string. The well-known duality between integer radius deformations and orbifolding suggests that at least locally and for integer β , refinement can also be given a more geometric interpretation in terms of a \mathbb{Z}_β orbifold. Namely, one may view the refinement in the B-model (for fixed integer β) near a conifold point in moduli space effectively as a replacement of the conifold singularity with an A_β singularity.

There is an apparent puzzle in this proposed orbifold interpretation. In the orbifold case, we have only an anti-selfdual background, so how do the coefficients $\Psi(\beta)$ arise then? Well, the answer is relatively simple. Under the \mathbb{Z}_β action, we do not have just one, but β massless hypermultiplets contributing to the leading coefficient

of (3.13). This is reflected in the fact that we can decompose the above Schwinger integral representation as (*cf.*, [37])

$$\mathcal{F}_{c=1}(\epsilon_1, \epsilon_2; t_c) = \sum_{n=0}^{\beta-1} \mathcal{F}_{c=1}(\epsilon_1, -\epsilon_1; t_n),$$

with $t_n := t_c - n\epsilon_2 - (\epsilon_1 + \epsilon_2)/2$. Hence, the heart of the proposed orbifold interpretation for integer β lies in the fact that we can trade a single massless hypermultiplet in the corresponding Ω -background for β massless hypermultiplets in an anti-selfdual background.

It is well known that the coefficients $\Psi^{(2n-2)}(1)$ correspond to the virtual Euler characteristic of the moduli space of complex curves of genus n , and one might ask for a similar interpretation for general β . As observed in [38, 39], up to a shift the coefficients for general β in fact match with the parameterized Euler characteristic interpolating between the virtual Euler characteristic of the moduli space of real and complex curves proposed in [40] (see also [41] for a more detailed discussion of this correspondence). The $c = 1$ string orbifold interpretation sketched above now allows us to conjecture a geometric interpretation of this parameterized Euler characteristic for integer β . Namely, it should correspond to the virtual Euler characteristic of genus n curves with a \mathbb{Z}_β action.

These local facts lead to the important question whether there exists as well a purely geometric interpretation of the refined partition function at large volume. That is, we should ask if there exists a target space \widehat{X}_β such that the refined free energy corresponds to the count of maps

$$\Sigma^{(g)} \rightarrow \widehat{X}_\beta. \quad (4.1)$$

Naively, one would suspect that \widehat{X}_β corresponds to a free \mathbb{Z}_β orbifold of the original Calabi-Yau X (for integer β). In particular, the only visible effect of such a free orbifold on the level of the holomorphic anomaly equations would be a mere change of boundary conditions, as we observed for refinement.

Indeed, one can find for specific models and values of β concrete proof that the refined partition function is dual to the usual topological string on an orbifold of the original geometry. The simplest example has been already given in [11], where it was observed that the quotient of local $\mathbb{P}^1 \times \mathbb{P}^1$ by its obvious \mathbb{Z}_2 symmetry equals the refined partition function on the original geometry at $\beta = 2$. A similar observation can

be made for orbifolding local \mathbb{P}^2 by its cyclic \mathbb{Z}_3 symmetry, which corresponds to the refined partition function at $\beta = 3$ [18].

One should note that the a priori undetermined function $\kappa(\beta)$ leaves the freedom for different analytic continuations of the \mathbb{Z}_β symmetry of the conifold to large volume. The necessity of such an ambiguity is intuitively clear, as there might exist at large volume differently acting symmetries, which still yield under analytic continuation the same leading singular behavior at the conifold point. For instance, there might be differently acting \mathbb{Z}_2 orbifolds at large volume, which, due to the high symmetry of the conifold, all possess the same coefficients $\Psi^{(n)}(2)$ (the massless hypermultiplet does not care which symmetry it feels).

In general, however, it is far from clear how, if at all, the \mathbb{Z}_β symmetry of the conifold point in moduli space translates to the large volume regime (*i.e.*, is globally preserved). In the above two examples, the correspondence between the refined topological string for particular values of β to the usual topological string on a different (orbifold) background could be argued to be a consequence of the large global symmetry group, and therefore somewhat accidental. This still leaves open the possibility for the existence of a new classical target space \widehat{X}_β , which could be obtained for instance in the case of β integer as a suitable partial compactification of an orbifold of the conifold.

5 Wave-function interpretation

The special geometry relation (3.5) between the flat coordinate a and the magnetic dual a_D is identical to the relation between canonically conjugate variables (p, q) of a classical integrable system. In particular, comparison with the Hamilton-Jacobi equation $H(q, \frac{\partial S}{\partial q}) = 0$ shows that in this interpretation, the prepotential $\mathcal{F}^{(0)}(a)$ should be identified with Hamilton's principal function (or classical action) effecting the canonical transformation to the action-angle variables.

Consider now the full perturbative partition function Z expanded as a series in g_s , as in (3.2). We have

$$Z = f(a, g_s, \beta) e^{\frac{1}{g_s^2} \int a_D da}, \quad (5.1)$$

with $f(a, g_s, \beta)$ some regular series in g_s^2 . This expansion shows that the partition function should be interpreted as a WKB-type wavefunction in a semi-classical approximation to a quantization of the original hamiltonian system.

In this context, it is important to keep in mind that quantization is intrinsically

ambiguous, *i.e.*, in general one cannot associate a unique quantum operator \hat{H} to a classical Hamiltonian H . This is most clearly apparent in the ordering ambiguities that plague the lifting of functions of the phase-space coordinates to quantum-mechanical operators. While the semi-classical terms are universal, the higher order terms in the expansion (5.1) are sensitive to these ambiguities.

The holomorphic anomaly plays an interesting role in this so-called “wavefunction interpretation” of the topological partition function. In fact, as pointed out by Witten [7], the holomorphic anomaly equation simply expresses the change of the wavefunction under a change of polarization of the underlying classical system (*i.e.*, the separation of the canonical coordinates into position and momentum variables). Above, we considered a real polarization, but complex polarizations are natural as well. In the sense of this wave-function interpretation, the topological partition function Z is a representation of the true ground state for a particular choice of polarization.

Witten’s original proposal was made for the partition function of topological strings (for which the holomorphic anomaly was first discovered), but given the results of [8, 11] that we have reviewed above, the interpretation is very natural in the context of gauge theory as well (*cf.*, [42]). An interesting consequence of the fact that the holomorphic anomaly equation is insensitive to the deformation parameter β is that the refined partition function corresponds to a family of quantum states with the same semi-classical expansion. On the other hand, it remains unclear how to determine the additional conditions that would select the topological partition function as the unique ground state of the system. To our knowledge, the wavefunction interpretation has not been successfully exploited for fixing the holomorphic ambiguity.

Via the AGT conjecture (for which there is now substantial evidence, as reviewed elsewhere in this volume), quantum Liouville theory provides an answer to the quantization problem that is in principle independent from the relation to topological strings, and has the advantage of being algorithmic. Yet another approach to the quantization problem are the so-called topological recursions of Eynard-Orantin [43]. A detailed comparison between these various schemes remains an interesting avenue for further research.

6 Outlook

We see two major open problems whose solution would constitute significant progress. The first, more technical in nature, is the direct and explicit calculation of the gauge theory partition function at strong coupling either via instanton counting or CFT. The results reviewed here, specifically the holomorphic limit of the $\mathcal{F}^{(g)}$ as $\bar{t}_D = 0$, $t_D \rightarrow 0$, provide a benchmark for such a calculation. A first indication that the 2d-4d relation holds beyond weak coupling has been found at hand of an explicit example recently in [26]. However, the general picture is far from clear. The perhaps most closely related work from a CFT point of view is [44]. Ultimately, a simple state like construction, as in [45], for the strongly coupled expansion would be desirable.

The second major open problem is to obtain a better understanding of the deformation parameter β in the topological string context. The core question is whether the deformation really involves a new world-sheet theory (for instance, with a second string coupling, in case the extra parameter is viewed as an infinitesimal coupling constant), or whether it might be sufficient to view the β -degree of freedom entirely as a geometric deformation of the target space of the usual topological string. In this review, we somewhat focussed on the latter point of view.

Though we did not discuss them in this review, it has to be mentioned that there are several explicit proposals in the literature [46, 47, 28, 15], supporting the possibility for an actual world-sheet interpretation with two infinitesimal coupling constants. In the mathematical formulation of the perturbative topological string (Gromov-Witten theory), this might involve a sort of refined count of holomorphic maps [9, 28]

$$\Sigma^{(g_1, g_2)} \rightarrow X, \tag{6.1}$$

with worldsheets Σ of genus $g = g_1 + g_2$, carrying an additional \mathbb{Z}_2 valued decoration of the handles.

Finally, another proposed interpretation involves replacing the B-model geometry by a sort of “quantum geometry” \tilde{Y}_q (encoding one of the parameters in a suitable parameterization). In this approach the tree-level special geometry depends explicitly on the extra parameter [48, 17]. This is analogous to the replacement of the spectral curve with a “quantum” spectral curve in the β -deformed matrix models reviewed in chapter [V:4], and the “quantization” of Seiberg-Witten theory outlined in chapter [V:11].

One should note that the quantum geometry \tilde{Y}_q is not directly related to the wave-function interpretation of the partition function discussed in section 5 (at least the precise relation is not known). While in the former we quantize the underlying curve, in the latter we quantize the periods. In this sense we can understand the refined topological string also as a double quantization. For \tilde{Y}_q the ordinary special geometry relation is lifted to a so-called quantum special geometry relation and the ordinary periods to quantum periods (depending on the extra parameter). Quantizing similarly as in section 5 the quantum periods, will again lead to the holomorphic anomaly equation, now for the (double) quantum states.

While neither proposal is entirely convincing in the present form, a possible connection between the two proposed target space deformations might even be more tantalizing. Optimistically, it might indicate a new type of classical-quantum duality relating the topological string on \hat{X}_β with that on \tilde{Y}_q . If in addition a refined topological string exists, in the sense of a deformed world-sheet theory, this duality would extend to a triality. This, at least, is the inspiration that we take away.

Acknowledgments We would like to thank J. Teschner for the invitation to participate in this joint review effort, his hard work and patience. We thank all other contributors for their valuable comments and input. The work of D.K. has been supported in part by a Simons fellowship, the Berkeley Center for Theoretical Physics and the National Research Foundation of Korea Grant No. 2012R1A2A2A02046739. The research of J.W. is supported in part by an NSERC discovery grant and a Tier II Canada Research Chair.

References

In this volume

- [V:1] D. Gaiotto, “Families of $N = 2$ field theories”.
- [V:2] A. Neitzke, “Hitchin systems in $\mathcal{N} = 2$ field theory”.
- [V:3] Y. Tachikawa, “A review on instanton counting and W-algebras”.
- [V:4] K. Maruyoshi, “ β -deformed matrix models and the 2d/4d correspondence.”
- [V:7] S. Gukov, “Surface Operators.”
- [V:11] J. Teschner, “Supersymmetric gauge theories, quantization of \mathcal{M}_{flat} , and Liouville theory.”

Other

- [1] N. A. Nekrasov, “Seiberg-Witten prepotential from instanton counting,” Adv. Theor. Math. Phys. **7** (2004) 831 [hep-th/0206161].
- [2] S. Kachru and C. Vafa, “Exact results for N=2 compactifications of heterotic strings,” Nucl. Phys. B **450**, 69 (1995) [hep-th/9505105]. S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, “Nonperturbative results on the point particle limit of N=2 heterotic string compactifications,” Nucl. Phys. B **459**, 537 (1996) [hep-th/9508155]. S. H. Katz, A. Klemm and C. Vafa, “Geometric engineering of quantum field theories,” Nucl. Phys. B **497**, 173 (1997) [hep-th/9609239]. S. Katz, P. Mayr and C. Vafa, “Mirror symmetry and exact solution of 4-D N=2 gauge theories: 1.,” Adv. Theor. Math. Phys. **1**, 53 (1998) [hep-th/9706110].
- [3] S. Hellerman, D. Orlando and S. Reffert, “String theory of the Omega deformation,” JHEP **1201**, 148 (2012) [arXiv:1106.0279 [hep-th]].
- [4] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, “Topological amplitudes in string theory,” Nucl. Phys. B **413** (1994) 162 [hep-th/9307158].
- [5] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes,” Commun. Math. Phys. **165** (1994) 311 [hep-th/9309140].
- [6] M. A. Shifman and A. I. Vainshtein, “Solution of the Anomaly Puzzle in SUSY Gauge Theories and the Wilson Operator Expansion,” Nucl. Phys. B **277**, 456 (1986) [Sov. Phys. JETP **64**, 428 (1986)] [Zh. Eksp. Teor. Fiz. **91**, 723 (1986)].
- [7] E. Witten, “Quantum background independence in string theory,” hep-th/9306122.
- [8] M. -x. Huang and A. Klemm, “Holomorphic Anomaly in Gauge Theories and Matrix Models,” JHEP **0709** (2007) 054 [hep-th/0605195].
- [9] T. J. Hollowood, A. Iqbal and C. Vafa, “Matrix models, geometric engineering and elliptic genera,” JHEP **0803**, 069 (2008) [hep-th/0310272]. A. Iqbal, C. Kozcaz and C. Vafa, “The Refined topological vertex,” JHEP **0910**, 069 (2009) [hep-th/0701156].

- [10] S. Gukov, A. S. Schwarz and C. Vafa, “Khovanov-Rozansky homology and topological strings,” *Lett. Math. Phys.* **74**, 53 (2005) [hep-th/0412243].
- [11] D. Krefl and J. Walcher, “Extended Holomorphic Anomaly in Gauge Theory,” *Lett. Math. Phys.* **95** (2011) 67 [arXiv:1007.0263 [hep-th]].
- [12] M. -x. Huang and A. Klemm, “Direct integration for general Ω backgrounds,” arXiv:1009.1126 [hep-th].
- [13] D. Krefl and J. Walcher, “Shift versus Extension in Refined Partition Functions,” arXiv:1010.2635 [hep-th].
- [14] D. Krefl and J. Walcher, unpublished (2010)
- [15] M. -x. Huang, A. -K. Kashani-Poor and A. Klemm, “The Ω deformed B-model for rigid $\mathcal{N} = 2$ theories,” *Annales Henri Poincare* **14**, 425 (2013) [arXiv:1109.5728 [hep-th]].
- [16] work in progress
- [17] M. Aganagic, M. C. N. Cheng, R. Dijkgraaf, D. Krefl and C. Vafa, “Quantum Geometry of Refined Topological Strings,” *JHEP* **1211** (2012) 019 [arXiv:1105.0630 [hep-th]].
- [18] D. Krefl, unpublished (2012)
- [19] A. Klemm, “On the geometry behind N=2 supersymmetric effective actions in four-dimensions,” In **Trieste 1996, High energy physics and cosmology** 120-242 [hep-th/9705131].
- [20] P. Mayr, “Geometric construction of N=2 gauge theories,” *Fortsch. Phys.* **47**, 39 (1999) [hep-th/9807096].
- [21] M. Kontsevich, “Enumeration of rational curves via Torus actions,” hep-th/9405035.
- [22] M. Aganagic, A. Klemm, M. Marino and C. Vafa, “The Topological vertex,” *Commun. Math. Phys.* **254** (2005) 425 [hep-th/0305132].
- [23] A. Klemm, M. Marino and S. Theisen, “Gravitational corrections in supersymmetric gauge theory and matrix models,” *JHEP* **0303** (2003) 051 [hep-th/0211216].
- [24] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Holomorphic anomalies in topological field theories,” *Nucl. Phys. B* **405** (1993) 279 [hep-th/9302103].
- [25] L. F. Alday, D. Gaiotto and Y. Tachikawa, “Liouville Correlation Functions from Four-dimensional Gauge Theories,” *Lett. Math. Phys.* **91** (2010) 167 [arXiv:0906.3219 [hep-th]].
- [26] D. Krefl, “Penner Type Ensemble for Gauge Theories Revisited,” arXiv:1209.6009 [hep-th].
- [27] D. Krefl and S. -Y. D. Shih, “Holomorphic Anomaly in Gauge Theory on ALE space,” arXiv:1112.2718 [hep-th].
- [28] A. Prudenziati, “Double genus expansion for general Ω background,” arXiv:1204.2322 [hep-th].
- [29] T. Okuda and V. Pestun, “On the instantons and the hypermultiplet mass of N=2* super Yang-Mills on S^4 ,” *JHEP* **1203**, 017 (2012) [arXiv:1004.1222 [hep-th]].

- [30] J. Walcher, “Extended holomorphic anomaly and loop amplitudes in open topological string,” Nucl. Phys. B **817** (2009) 167 [arXiv:0705.4098 [hep-th]].
- [31] J. Walcher, “Evidence for Tadpole Cancellation in the Topological String,” Commun. Num. Theor. Phys. **3** (2009) 111 [arXiv:0712.2775 [hep-th]].
- [32] M. -x. Huang, A. Klemm and S. Quackenbush, “Topological string theory on compact Calabi-Yau: Modularity and boundary conditions,” Lect. Notes Phys. **757** (2009) 45 [hep-th/0612125].
- [33] D. Ghoshal and C. Vafa, “ $C = 1$ string as the topological theory of the conifold,” Nucl. Phys. B **453** (1995) 121 [hep-th/9506122].
- [34] B. Haghighat, A. Klemm and M. Rauch, “Integrability of the holomorphic anomaly equations,” JHEP **0810** (2008) 097 [arXiv:0809.1674 [hep-th]].
- [35] C. Vafa, “A Stringy test of the fate of the conifold,” Nucl. Phys. B **447** (1995) 252 [hep-th/9505023].
- [36] D. J. Gross and I. R. Klebanov, “One-dimensional String Theory On A Circle,” Nucl. Phys. B **344** (1990) 475.
- [37] R. Gopakumar and C. Vafa, “Topological gravity as large N topological gauge theory,” Adv. Theor. Math. Phys. **2** (1998) 413 [hep-th/9802016].
- [38] D. Krefl and J. Walcher, “ABCD of Beta Ensembles and Topological Strings,” JHEP **1211**, 111 (2012) [arXiv:1207.1438 [hep-th]].
- [39] D. Krefl and A. Schwarz, “Refined Chern-Simons versus Vogel universality,” J. Geom. Phys. **74** (2013) 119 [arXiv:1304.7873 [hep-th]].
- [40] I. P. Goulden, J. L. Harer and D. M. Jackson, “A geometric parametrization for the virtual Euler characteristic of the moduli space of real and complex algebraic curves,” Trans. Am. Math. Soc. **353**, 4405 (2001)
- [41] N. Chair, “Generalized Penner model and the Gaussian beta ensemble,” Nucl. Phys. B, in press.
- [42] M. Aganagic, V. Bouchard and A. Klemm, “Topological Strings and (Almost) Modular Forms,” Commun. Math. Phys. **277**, 771 (2008) [hep-th/0607100].
- [43] B. Eynard and N. Orantin, “Invariants of algebraic curves and topological expansion,” Commun. Num. Theor. Phys. **1**, 347 (2007) [math-ph/0702045].
- [44] D. Gaiotto and J. Teschner, “Irregular singularities in Liouville theory and Argyres-Douglas type gauge theories, I,” JHEP **1212** (2012) 050 [arXiv:1203.1052 [hep-th]].
- [45] D. Gaiotto, “Asymptotically free $N=2$ theories and irregular conformal blocks,” arXiv:0908.0307 [hep-th].
- [46] I. Antoniadis, S. Hohenegger, K. S. Narain and T. R. Taylor, “Deformed Topological Partition Function and Nekrasov Backgrounds,” Nucl. Phys. B **838** (2010) 253 [arXiv:1003.2832 [hep-th]]. I. Antoniadis, I. Florakis, S. Hohenegger, K. S. Narain and

- A. Zein Assi, “Worldsheet Realization of the Refined Topological String,” Nucl. Phys. B **875** (2013) 101 [arXiv:1302.6993 [hep-th]]. I. Antoniadis, I. Florakis, S. Hohenegger, K. S. Narain and A. Zein Assi, “Non-Perturbative Nekrasov Partition Function from String Theory,” arXiv:1309.6688 [hep-th].
- [47] Y. Nakayama and H. Ooguri, “Comments on Worldsheet Description of the Omega Background,” Nucl. Phys. B **856** (2012) 342 [arXiv:1106.5503 [hep-th]].
- [48] A. Mironov and A. Morozov, “Nekrasov Functions and Exact Bohr-Zommerfeld Integrals,” JHEP **1004** (2010) 040 [arXiv:0910.5670 [hep-th]].